Numerical Simulations of Ionospheric Dynamo Model in a Non-uniform Magnetic Field

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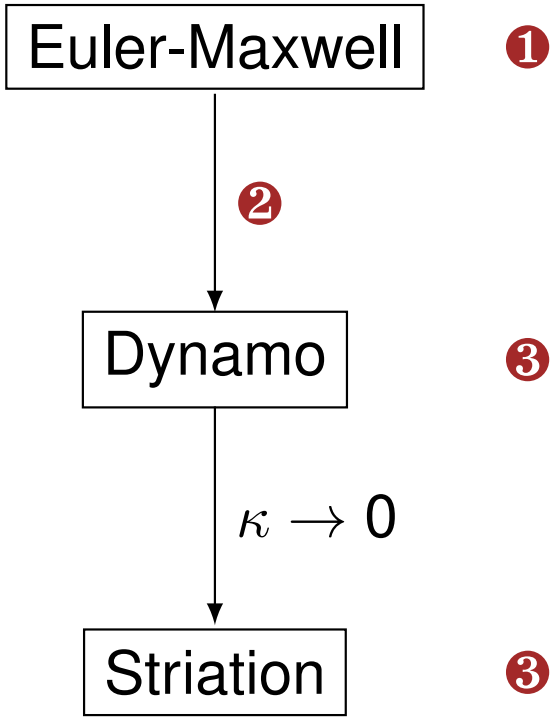
Outline

Outline

Ionospheric Dynamo Model I



Strategy

❶ Scaling

❷ Simplify the model by passing several dimensionless parameters to 0

❸ Pass to "Euler Potentials" coordinates system

Ionospheric Dynamo Model II

After ❶ and ❷, we have

Dynamo Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_i) = 0,$$
$$E + u_i \times B = \kappa \nu_i (u_i - u_n),$$
$$E + u_e \times B = -\kappa \nu_e (u_e - u_n),$$
$$\nabla \cdot j = 0,$$
$$\nabla \times E = 0,$$
$$\kappa j = \rho (u_i - u_e).$$

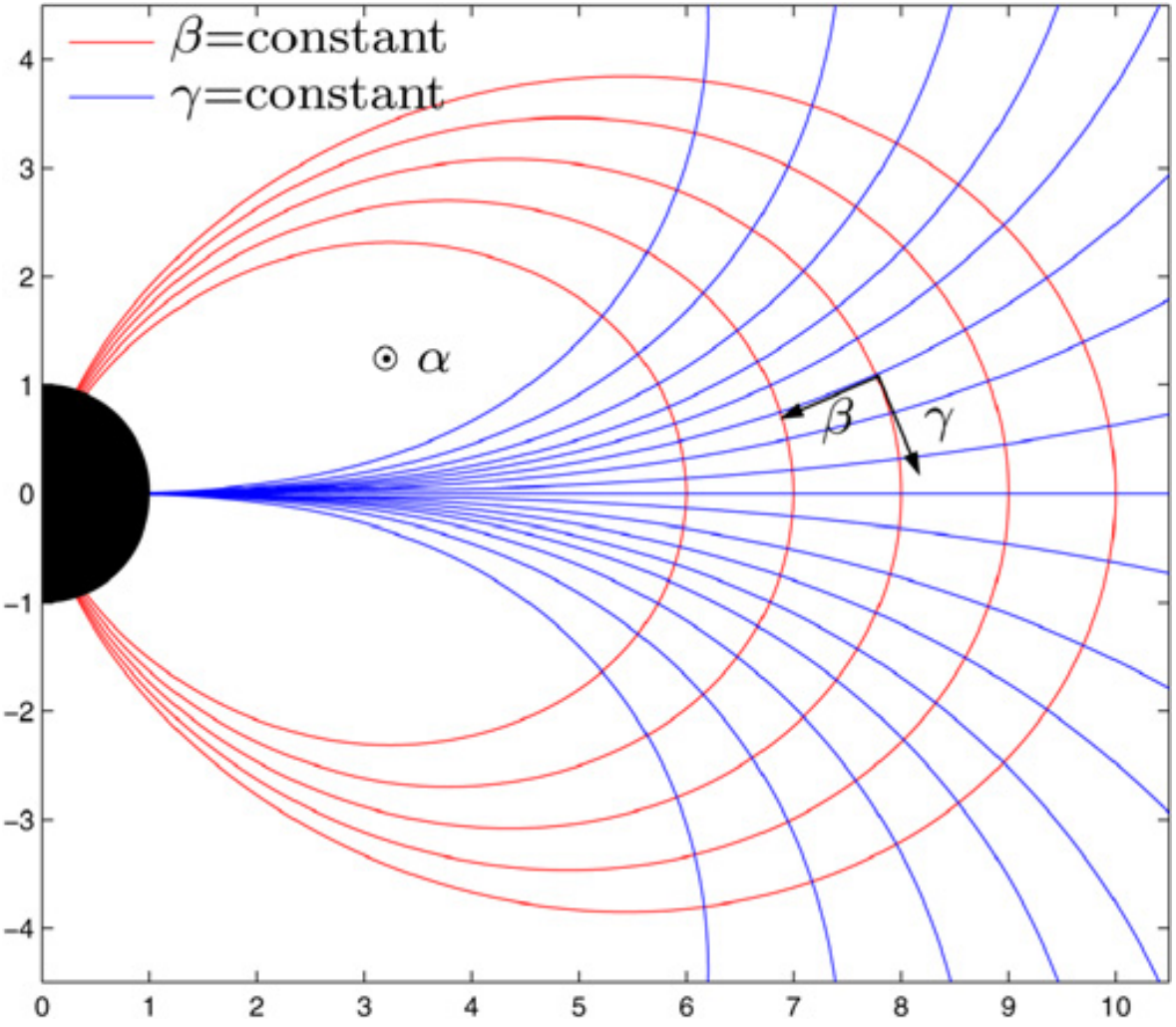
This model is valid in all the layers of ionosphere.

Notations

- ρ: density of plasma
- u_n: velocity of neutral wind
- u_e, u_i: electron and ion velocities
- ν_e, ν_i: e-n and i-n collision frequencies
- t: time
- B: magnetic field
- j: plasma current
- E: electric field
- κ: parameter measures the typical number of e-n or i-n collisions $\kappa \sim 10^{-4}$

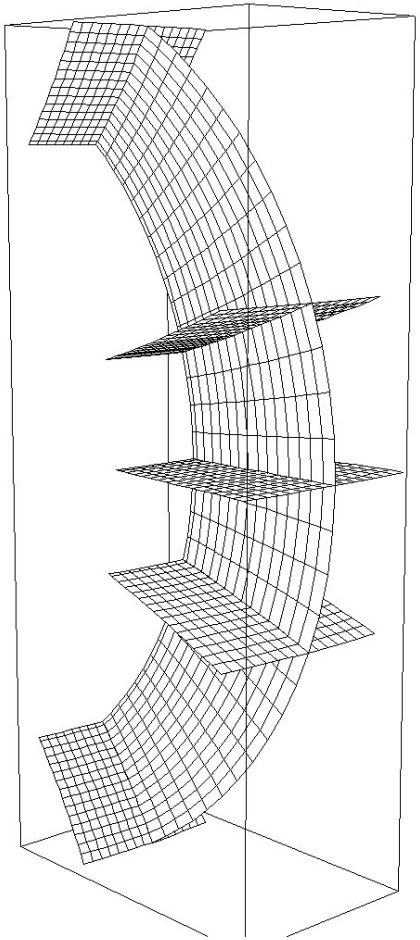
"Euler Potentials" Coordinates System I

Orthogonal coordinates (α, β, γ)



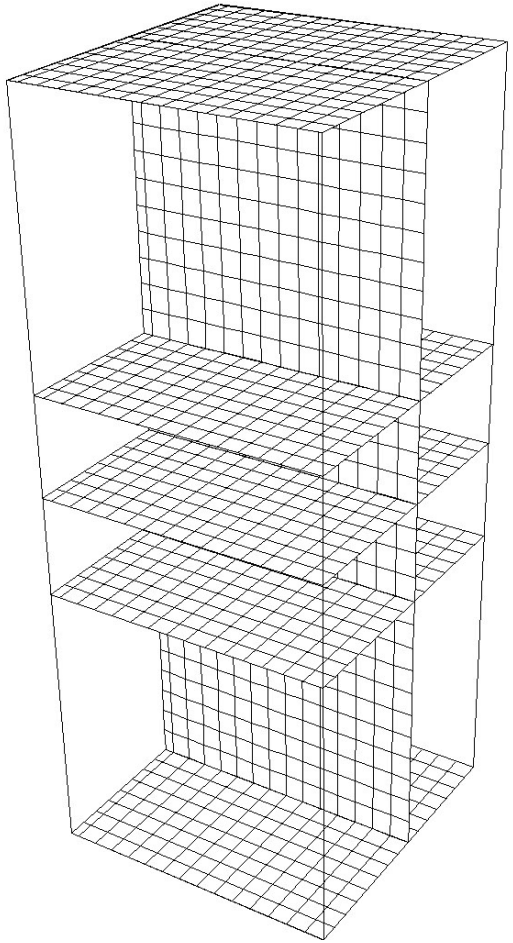
"Euler Potentials" Coordinates System II

Mesh in Cartesian coordinates



$\Longrightarrow$

Mesh in "Euler Potentials" coordinates



"Euler Potentials" Coordinates System III

In “Euler Potentials” coordinates system, we can recast the ionospheric dynamo system into a 3D transport equation

$$\frac{\partial w}{\partial t} + \nabla \cdot (vw) = 0, \tag{1}$$

paired with a 3D elliptic equation

$$-\nabla \cdot (\mathcal{A} \nabla \phi) = -\nabla \cdot J_n, \tag{2}$$

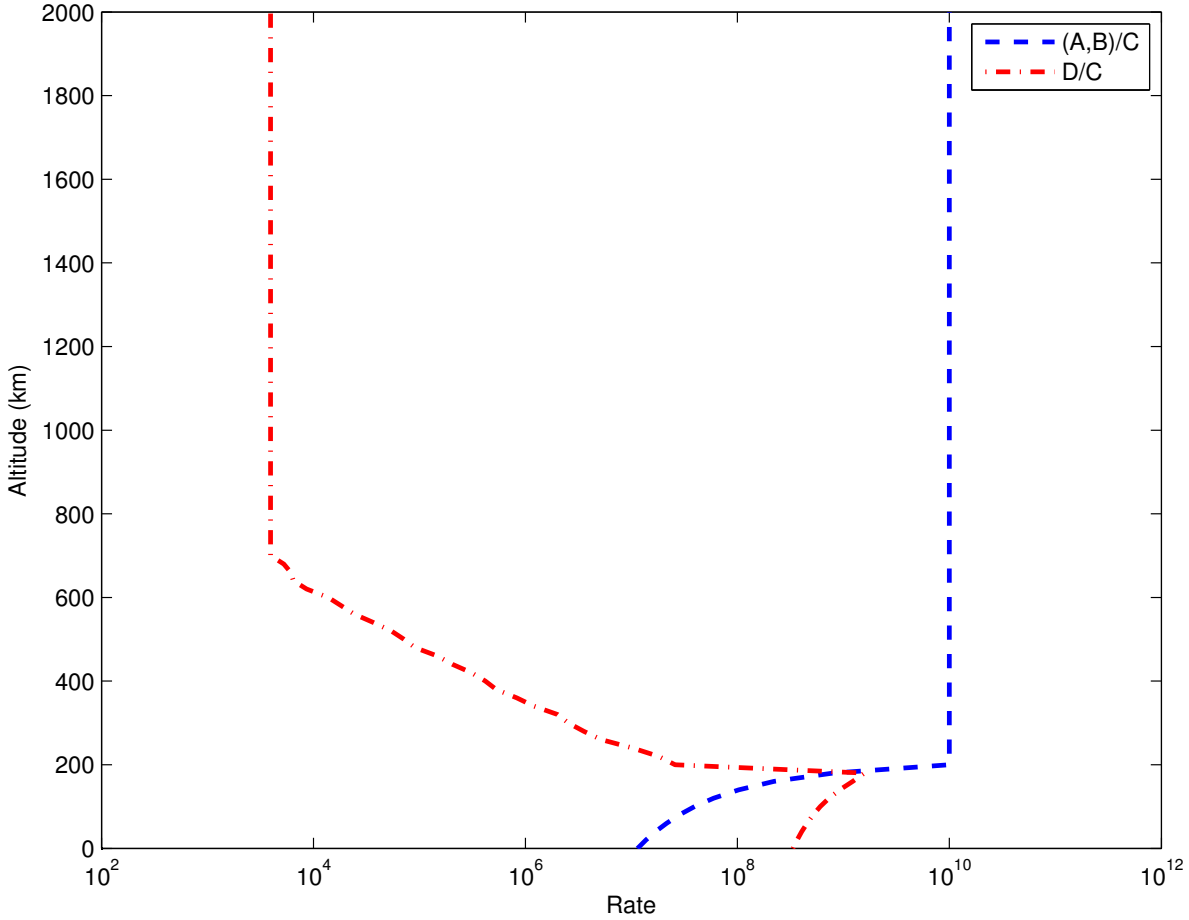
where the diffusion matrix is

$$\mathcal{A} = \begin{pmatrix} \mathrm{A} & -\mathrm{D} & 0 \\ \mathrm{D} & \mathrm{B} & 0 \\ 0 & 0 & \mathrm{C} \end{pmatrix}.$$

Anisotropy of Elliptic Equation

The magnitude of diffusion matrix coefficients

$$A, B \sim \frac{\nu_e}{(\kappa\nu_e)^2 + |B|^2}, \; D \sim \frac{\kappa\nu_e^2}{(\kappa\nu_e)^2 + |B|^2}, \; C \sim \frac{1}{\kappa^2\nu_i}.$$



Ionospheric Striation Model I

By passing κ to 0, we have

Striation Model

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$
$$E + u \times B = 0,$$
$$j \times B = \nu \rho (u - u_n),$$
$$\nabla \cdot j = 0,$$
$$\nabla \times E = 0,$$

In "Euler Potentials" coordinates, the striation model can be written as

- a 3D transport equation
- a 2D isotropic elliptic equation

where $\nu := \nu_i + \nu_e$, $u := u_i = u_e$.
The striation model is valid in high altitude where the collision frequency is weak.

Ionospheric Striation Model II

The 2D elliptic equation is given by

$$-\nabla \cdot (\mathcal{A}\nabla\phi) = -\nabla \cdot J_n, \tag{3}$$

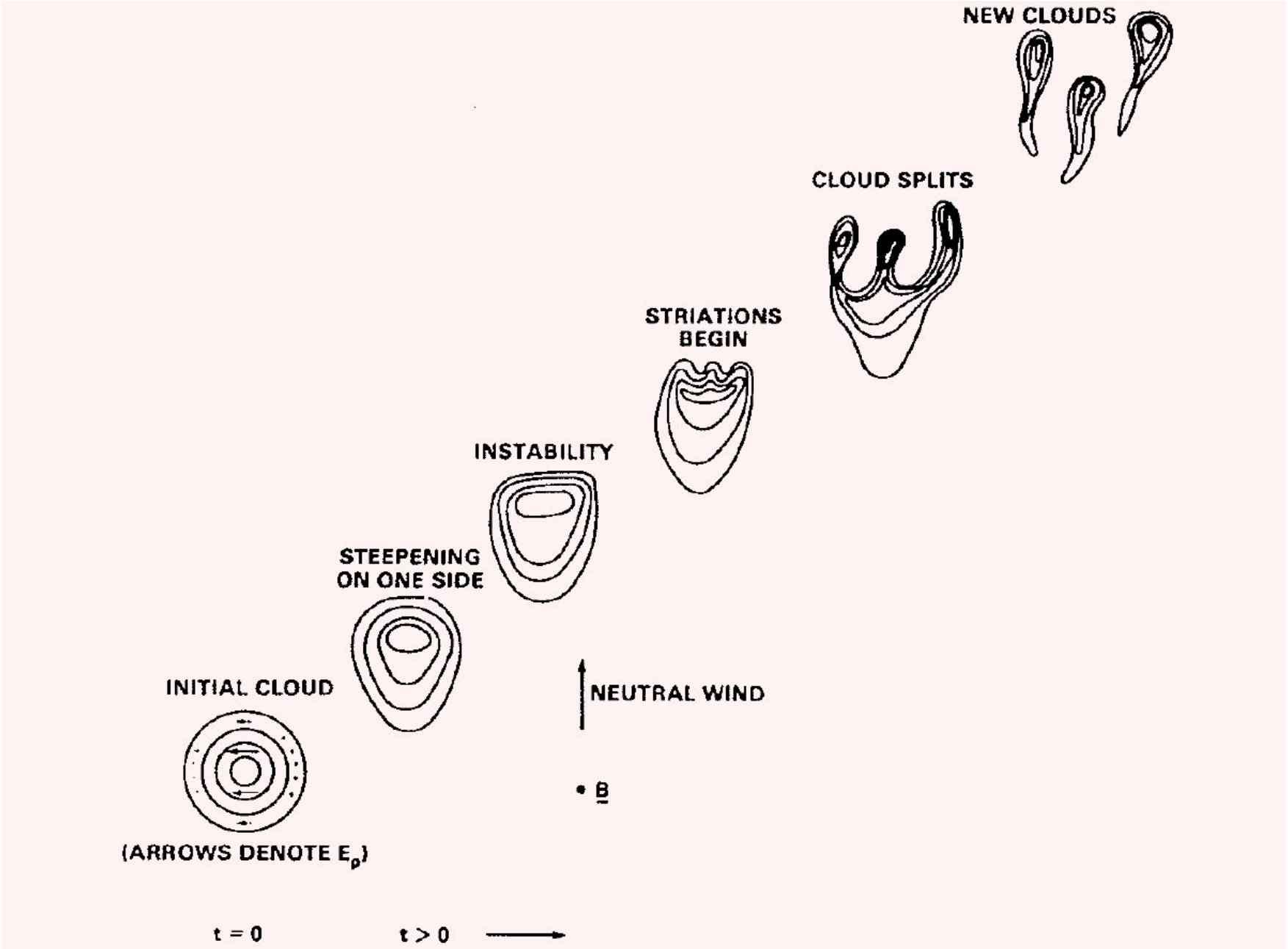
where the diffusion matrix is

$$\mathcal{A} = \begin{pmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{pmatrix},$$

with $\mathrm{A} = \int_{\gamma_{\min}}^{\gamma_{\max}} \rho\nu \frac{\mathrm{d}\gamma}{r^2 \sin^2\varphi |B|^4}$, $\mathrm{B} = \int_{\gamma_{\min}}^{\gamma_{\max}} \rho\nu \frac{r^2 \sin^2\varphi \mathrm{d}\gamma}{|B|^2}$.

Our goal is to simulate the plasma cloud on equator where the collision frequency is weak, so the information of the plasma cloud in terms A, B are too weak. For this, we have to use constant frequency in striation model.

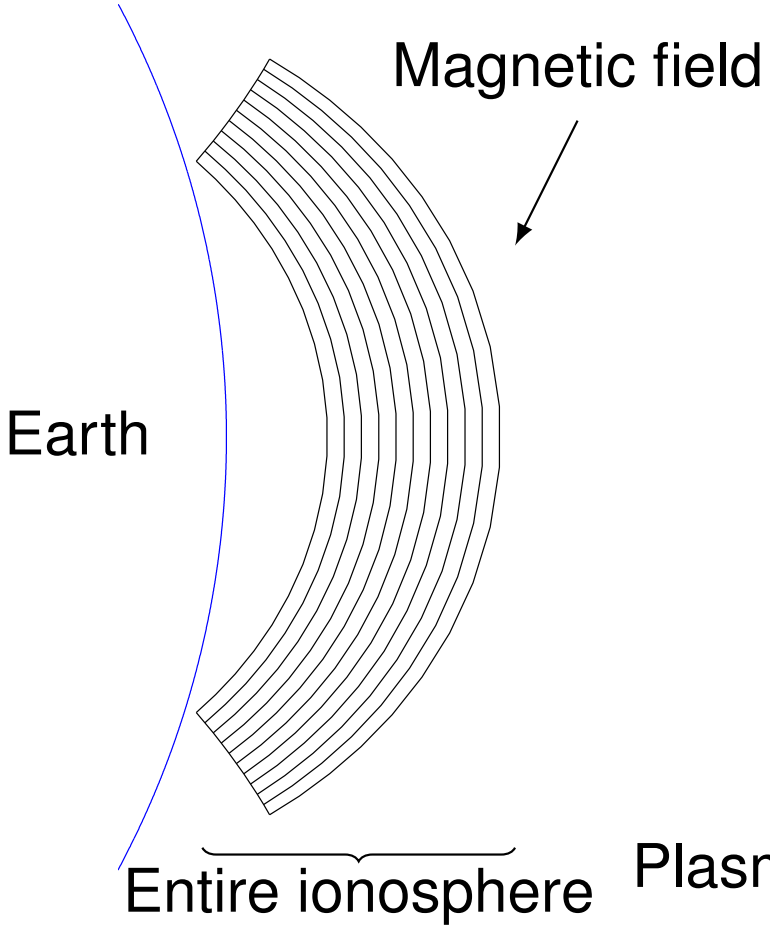
Striation Process

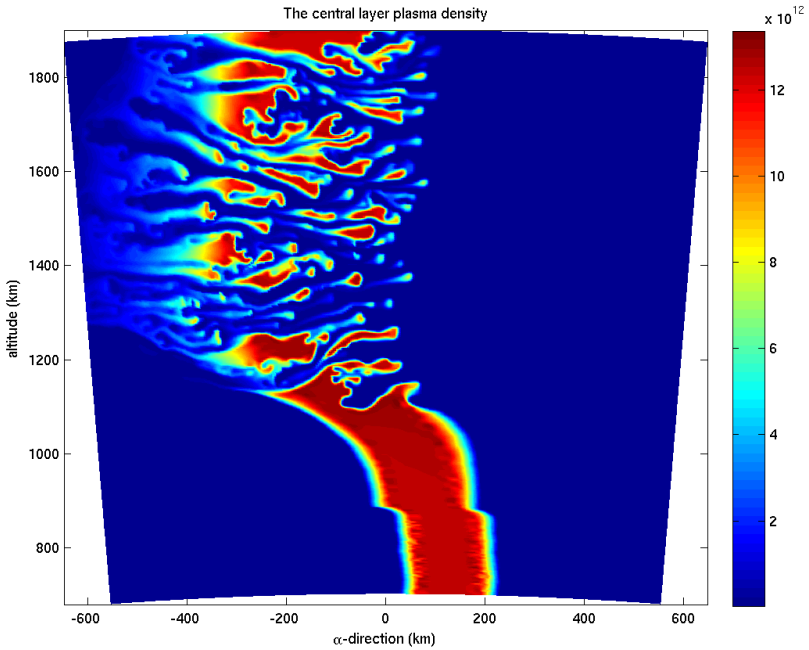


Outline

Computational Domain I

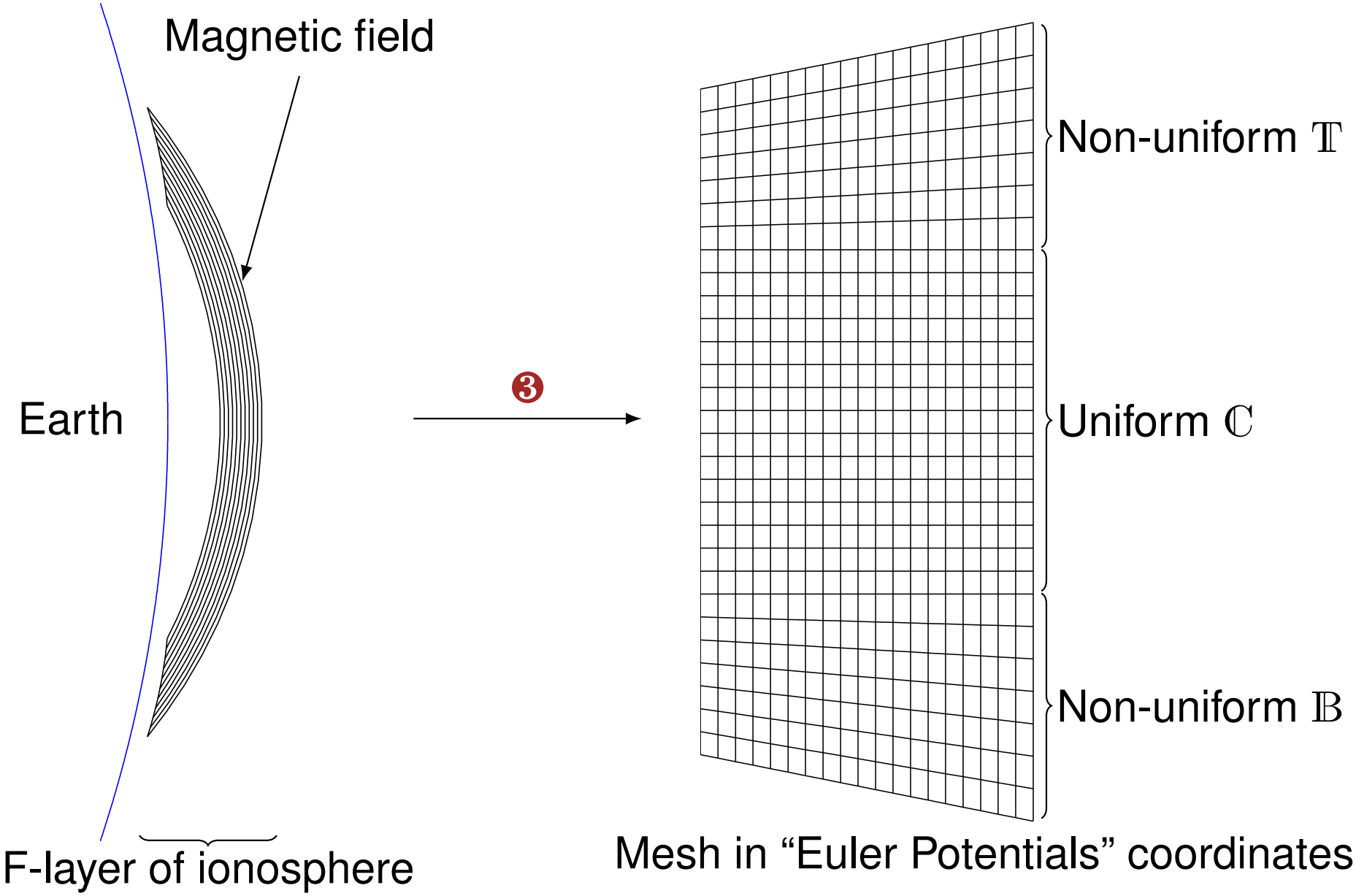
- C. Besse, J. Claudel, P. Degond, F. Deluzet, G. Gallice, C. Tessieras, *Numerical simulations of the ionospheric striation model in a non-uniform magnetic field*, Computer physics communications, 2006.





Plasma density in the central layer after $t = 2$h

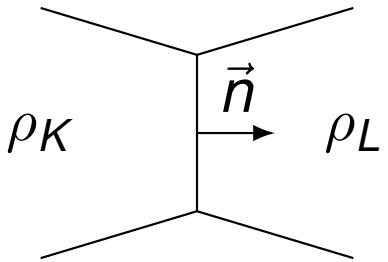
Computational Domain II



Computation of the Density

- Method A: Tridimensional finite volume method
  The numerical flux is defined as

$$F_{K|L} := \begin{cases} \text{mes}(K|L)\vec{v}\cdot\vec{n}\rho_K, & \text{if } \vec{v}\cdot\vec{n} \geq 0, \\ \text{mes}(K|L)\vec{v}\cdot\vec{n}\rho_L, & \text{if } \vec{v}\cdot\vec{n} < 0, \end{cases}$$



where $\vec{v}$ is the average velocity of surface $K|L$, $\vec{n}$ is its external normal.

- Method B: Domain decomposition method
  We decompose the computational domain into 3 parts.
  - In the top part $\mathbb{T}$ and the bottom part $\mathbb{B}$: the tridimensional finite volume method
  - In the center part $\mathbb{C}$: the Lie splitting method with SuperBee flux limiter scheme

Asymptotic-Preserving (AP) scheme I

Since the elliptic equation is anisotropic, we have to develop an AP scheme to solve it.

AP scheme

- S. JIN introduced AP scheme
- P. DEGOND, F. DELUZET, C. NEGULESCU gave an AP scheme for 2D elliptic problems with constant anisotropy

We extract a singularly perturbed problem from the real physical problem as follows

$$\text{(SP)} \begin{cases} -\dfrac{\partial}{\partial x}\left(\varepsilon A_\perp \dfrac{\partial \phi^\varepsilon}{\partial x}\right) - \dfrac{\partial}{\partial z}\left(A_z \dfrac{\partial \phi^\varepsilon}{\partial z}\right) = \varepsilon f, & \text{in } \Omega, \\ \phi^\varepsilon = 0, & \text{on } \partial\Omega_x \times \Omega_z, \\ \partial_z \phi^\varepsilon = 0, & \text{on } \Omega_x \times \partial\Omega_z, \end{cases} \tag{4}$$

where ε is constant between 0 and 1. The limit for $\varepsilon \to 0$ of problem (SP) is ill-posed.

Asymptotic-Preserving (AP) scheme II

We pick a new reformulation of AP scheme. We note

$$f = \bar{f} + f',$$

where $\bar{f} := \dfrac{1}{\mathrm{mes}(\Omega_z)} \displaystyle\int_{z_{\min}}^{z_{\max}} f \mathrm{d}z$, $f' := f - \bar{f}$. By integrating equation (4) along the axis z, we obtain an average equation

$$\begin{cases} -\dfrac{\partial}{\partial x}\left(\overline{A_\perp \dfrac{\partial \bar{\phi}}{\partial x}}\right) = \bar{f} + \dfrac{\partial}{\partial x}\left(\overline{A_\perp \dfrac{\partial \phi'}{\partial x}}\right), & \text{in } \Omega_x, \\ \bar{\phi} = 0, & \text{on } \partial\Omega_x. \end{cases} \tag{5}$$

Then by introducing the decomposition $\phi = \bar{\phi} + \phi'$ in equation (4), we get a fluctuation equation

$$\begin{cases} -\varepsilon \dfrac{\partial}{\partial x}\left(A_\perp \dfrac{\partial \phi'}{\partial x}\right) - \dfrac{\partial}{\partial z}\left(A_z \dfrac{\partial \phi'}{\partial z}\right) = \varepsilon f + \varepsilon \dfrac{\partial}{\partial x}\left(A_\perp \dfrac{\partial \bar{\phi}}{\partial x}\right), & \text{in } \Omega, \\ \phi' = 0, \text{ on } \partial\Omega_x \times \Omega_z, \\ \partial_z \phi' = 0, \text{ on } \Omega_x \times \partial\Omega_z, \\ \overline{\phi'} = 0, \text{ in } \Omega_x. \end{cases} \tag{6}$$

Asymptotic-Preserving (AP) scheme III

The equations (5)-(6) can hold the asymptotic preserving property, so they are well-posed for all $\varepsilon > 0$.

Finally, via the introduction of a Lagrange multiplier $\bar{p}$, we have the following AP-reformulation

$$(\mathrm{AP})\begin{cases} a_1(\phi', \psi') + b_1(\bar{p}, \psi') = f_1(\psi') - c_1(\bar{\phi}, \psi'), & \forall \psi' \in \mathbb{V}, \\ a_2(\bar{\phi}, \bar{\psi}) = f_2(\bar{\psi}) - c_2(\phi', \bar{\psi}), & \forall \bar{\psi} \in \mathbb{W}, \\ b_2(\phi', \bar{q}) = 0, & \forall \bar{q} \in \mathbb{W}. \end{cases} \tag{7}$$

In previous paper, the AP scheme is solved iteratively between average equation and fluctuation equation. We propose a discretization as follows

$$\begin{pmatrix} A_1 & C_1 & B_1 \\ C_2 & A_2 & 0 \\ B_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi' \\ \bar{\Phi} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}, \tag{8}$$

which solves the AP scheme directly.

Asymptotic-Preserving (AP) scheme IV

Advantages of new AP scheme

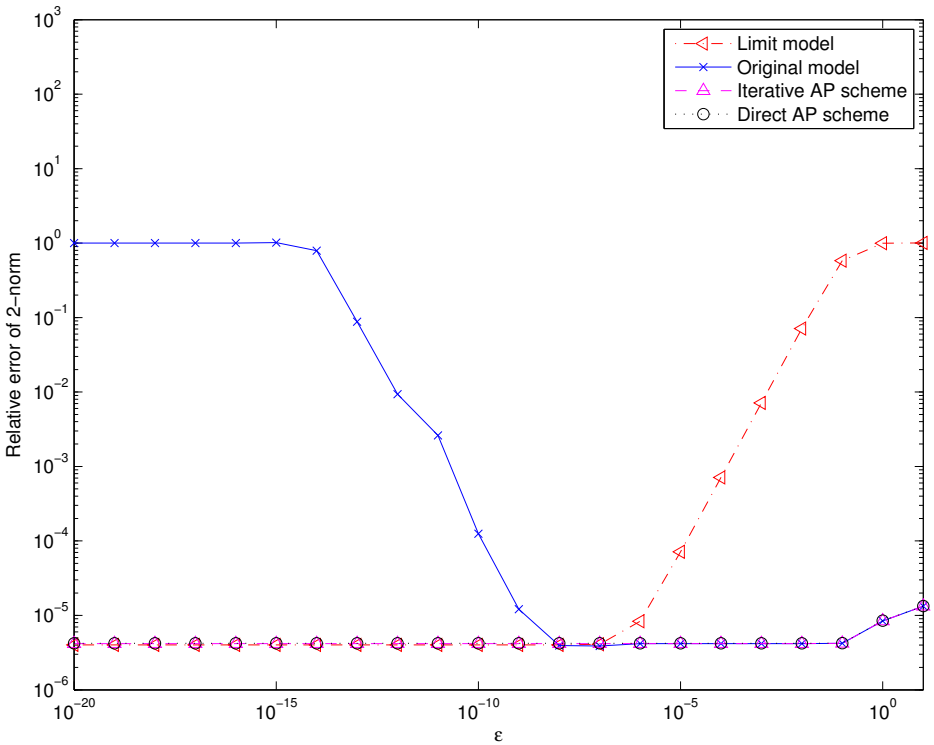
- The matrix (8) is sparser than class AP scheme
- The matrix (8) can be easily determined from the matrix of SP-problem
- The direct resolution is more efficient than the iterative resolution

To verify the new AP scheme, we impose a exact solution as follows

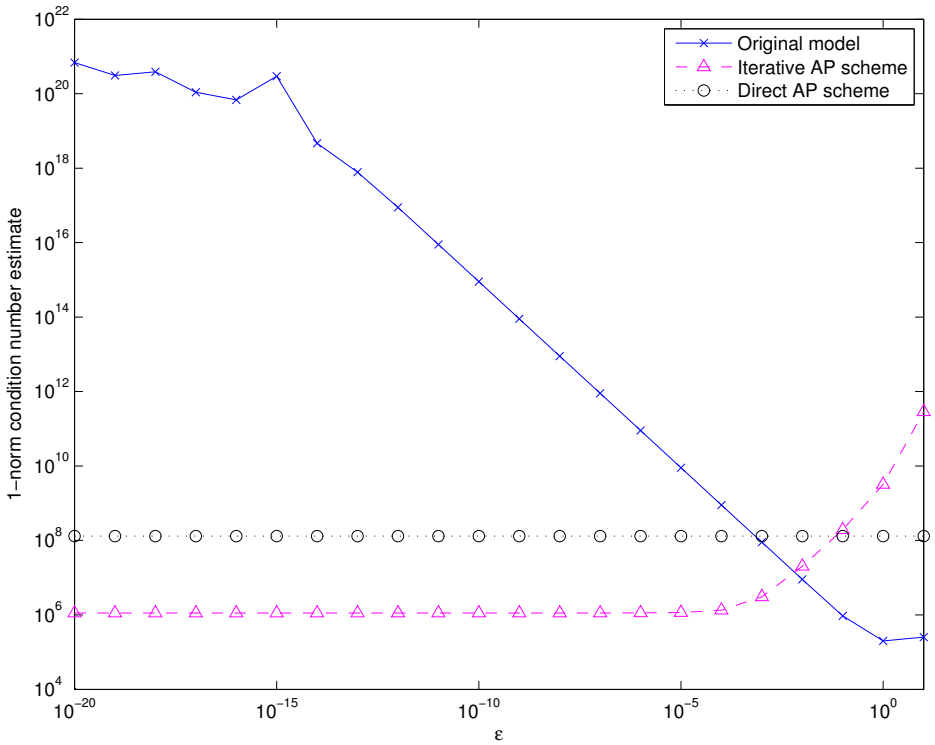
$$\phi_{\text{ext}} = \sin\left(\frac{2\pi}{L_x}x\right)\left(1 + \varepsilon\cos\left(\frac{2\pi}{L_z}z\right)\right)$$

By injecting ϕ_{ext} into (4), we obtain the right hand-side f. In the following test, we vary ε from 10^{-20} to 1.

Asymptotic-Preserving (AP) scheme V



Relative errors



Condition number estimations

Asymptotic-Preserving (AP) scheme VI

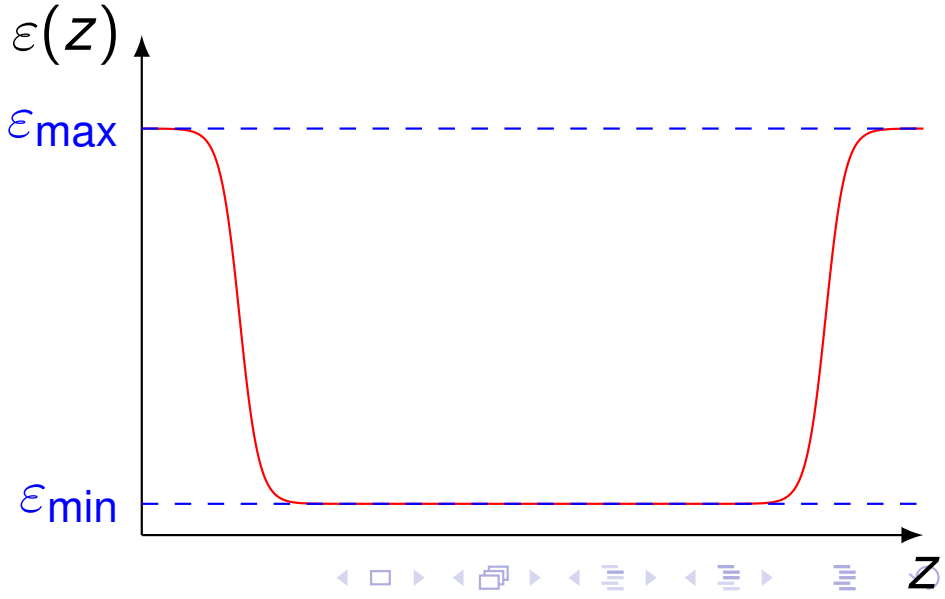
More commonly, ε is a variable, for example ε is a function of z. Thus the corresponding fluctuation equation shall be written as

$$-\frac{\partial}{\partial x}\left(A_\perp \frac{\partial \phi'}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{A_z}{\varepsilon}\frac{\partial \phi'}{\partial z}\right) = f + \frac{\partial}{\partial x}\left(A_\perp \frac{\partial \bar{\phi}}{\partial x}\right), \quad \text{in } \Omega.$$

And the term of Lagrange multiplier is

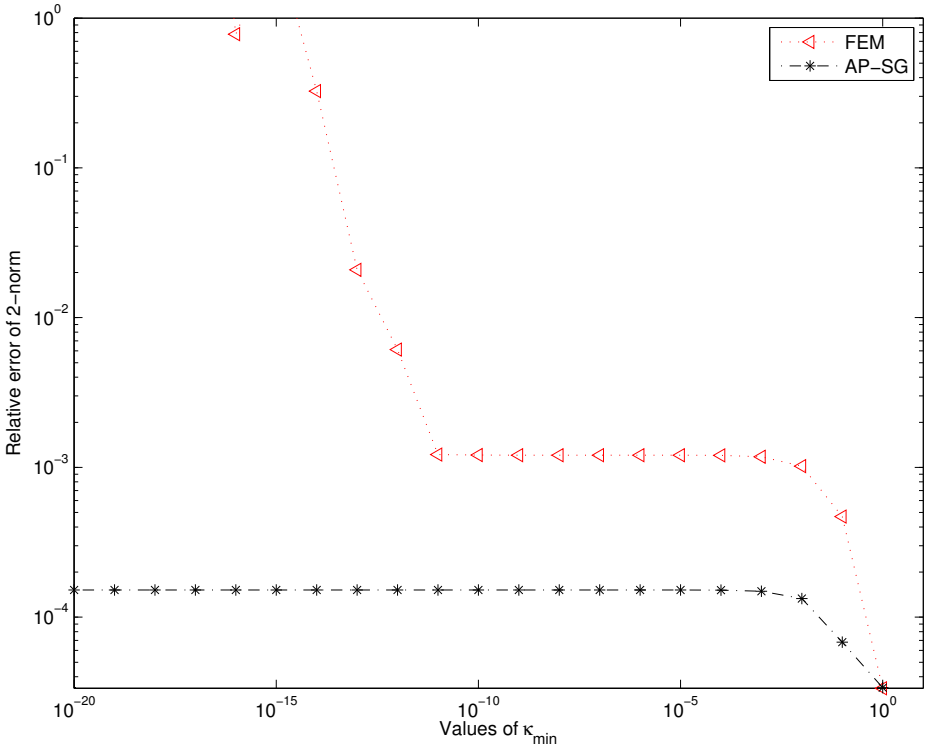
$$b_1(\bar{p}, \psi') := \int_0^{Lx} \bar{p}(x) \int_0^{Lz} \frac{1}{\varepsilon(z)} \psi'(x,z) \mathrm{d}z \mathrm{d}x.$$

To verify the efficacy of this AP scheme, we choose a function $0 < \varepsilon(z) \leq 1$ controlled by $\varepsilon_{\max}$ and $\varepsilon_{\min}$ to introduce variable anisotropy. Then we impose an exact solution to compare with the numerical solutions.

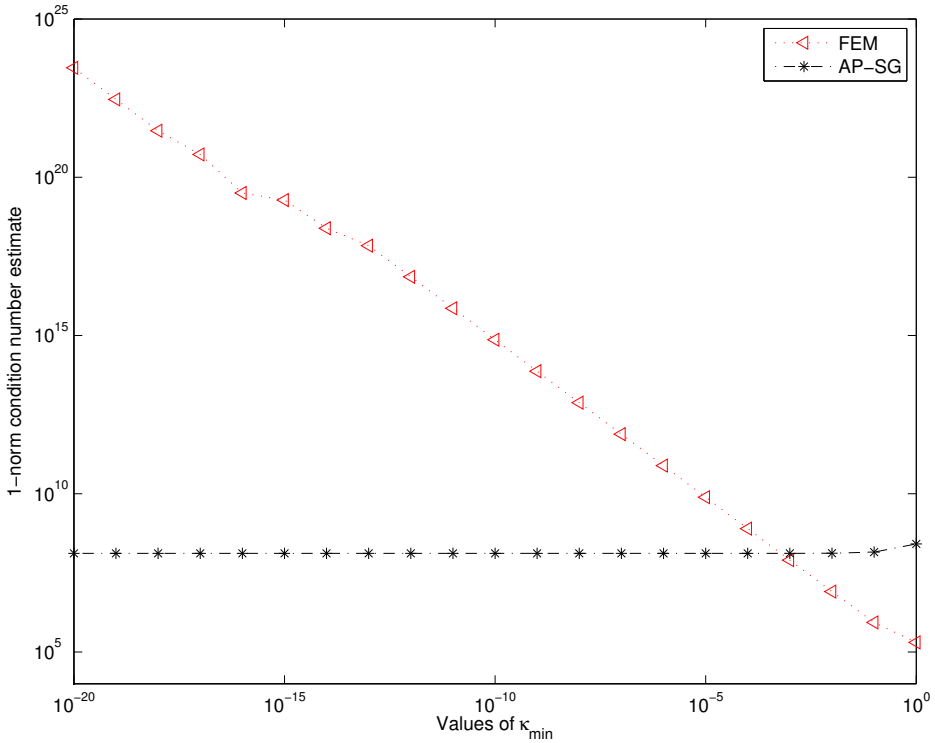


Asymptotic-Preserving (AP) scheme VII

To solve SP-problem, we use the finite element method and AP scheme respectively. Moreover, we note that there are large gradient in function $\varepsilon(z)$, thus we combine a Scharfetter-Gummel scheme with AP scheme.



Relative errors



Condition number estimations

Computation of the Potential

We rewrite the elliptic equation (2) as follows

$$-\nabla \cdot \left(\begin{pmatrix} \mathcal{A}_\perp & 0 \\ 0 & \mathcal{A}_\gamma \end{pmatrix} \begin{pmatrix} \nabla^\perp \phi \\ \partial_\gamma \phi \end{pmatrix} \right) = F, \tag{9}$$

where $\mathcal{A}_\perp := \begin{pmatrix} \mathsf{A} & -\mathsf{D} \\ \mathsf{D} & \mathsf{B} \end{pmatrix}$, $\mathcal{A}_\gamma := \mathsf{C}$, $F := -\nabla \cdot J_n$, $\nabla^\perp := \begin{pmatrix} \partial_\alpha \\ \partial_\beta \end{pmatrix}$.

Following the AP scheme in variable ε-case, we can develop the AP scheme for the equation (9), that is:

- Average equation

$$\begin{cases} -\nabla^\perp \cdot \left(\bar{\mathcal{A}}_\perp \nabla^\perp \bar{\phi} \right) = \bar{F} + \nabla^\perp \left(\overline{\mathcal{A}_\perp \nabla^\perp \phi'} \right), \\ \text{B. C.} \end{cases} \tag{10}$$

- Fluctuation equation

$$\begin{cases} -\nabla \cdot \left(\mathcal{A} \nabla \phi' \right) = F + \nabla^\perp \left(\mathcal{A}_\perp \nabla^\perp \bar{\phi} \right), \\ \bar{\phi}' = 0, \\ \text{B. C.} \end{cases} \tag{11}$$

Boundary Conditions

- Transport equation
  We take the Neumann homogeneous condition for all Ω
- Elliptic equation
  - Natural boundary conditions

$$\begin{cases} \mathcal{A}\nabla\phi\cdot\overrightarrow{n} = 0, \text{ on } \partial\Omega_\alpha\times\Omega_\beta\times\Omega_\gamma\cup\Omega_\alpha\times\Omega_\beta\times\partial\Omega_\gamma, \\ \phi = 0, \text{ on } \Omega_\alpha\times\partial\Omega_\beta\times\Omega_\gamma, \end{cases} \tag{12}$$

    Existence and uniqueness of solution ✔
  - Physical boundary conditions

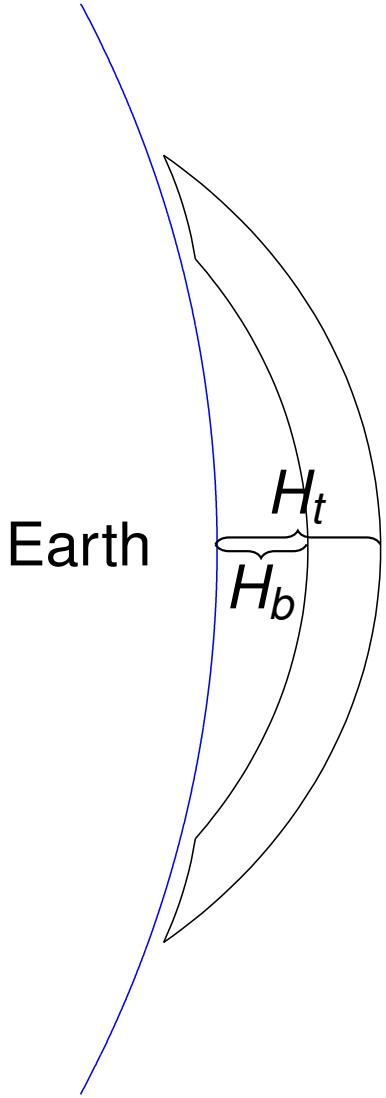
$$\begin{cases} \partial_\alpha\phi = 0, \text{ on } \partial\Omega_\alpha\times\Omega_\beta\times\Omega_\gamma, \\ \phi = 0, \text{ on } \Omega_\alpha\times\partial\Omega_\beta\times\Omega_\gamma, \\ \partial_\gamma\phi = 0, \text{ on } \Omega_\alpha\times\Omega_\beta\times\partial\Omega_\gamma \end{cases} \tag{13}$$

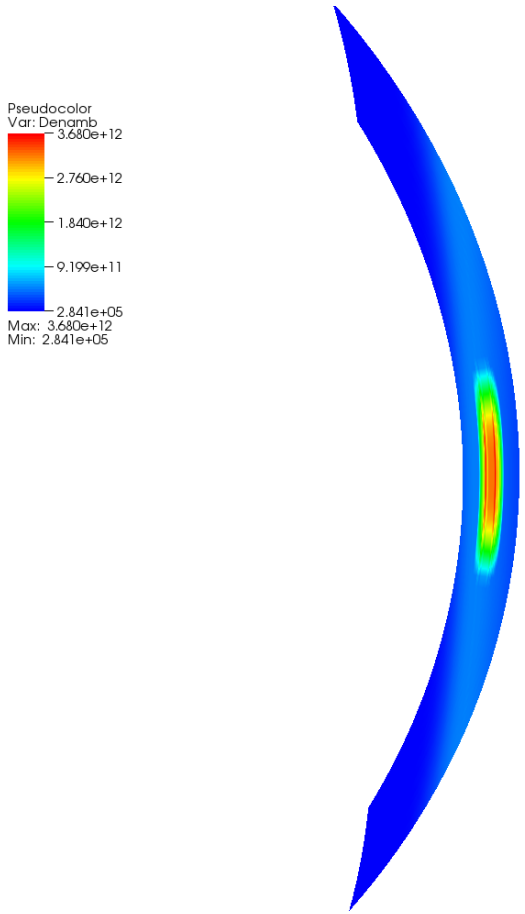
    Existence and uniqueness of solution ?

Outline

Initial Data

We use the “IRI” model for the initial data.

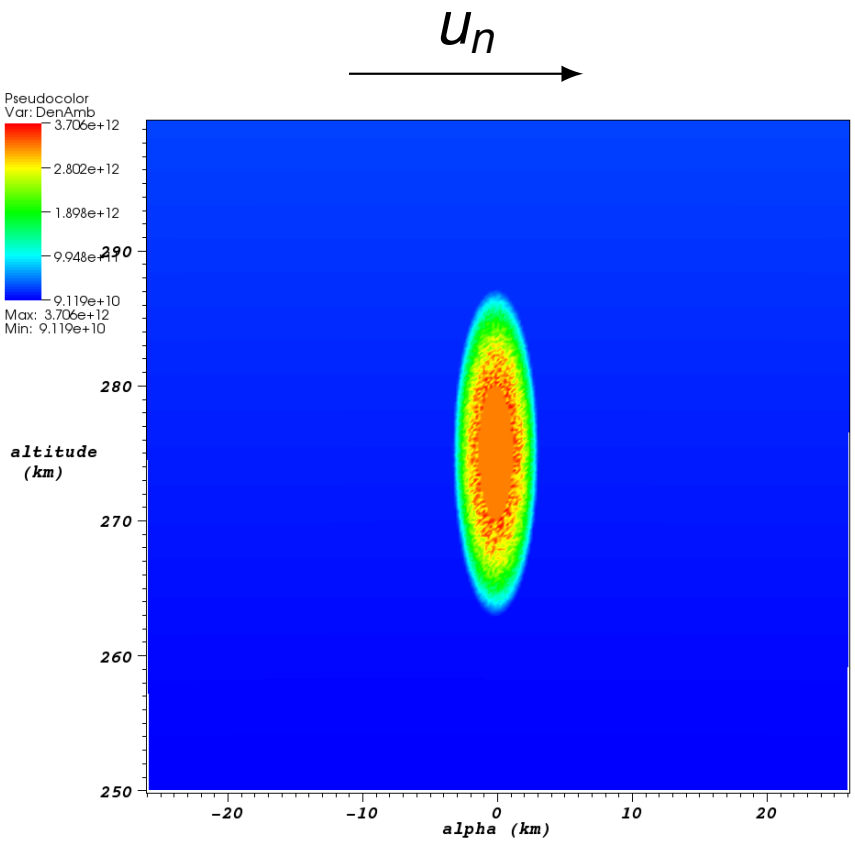


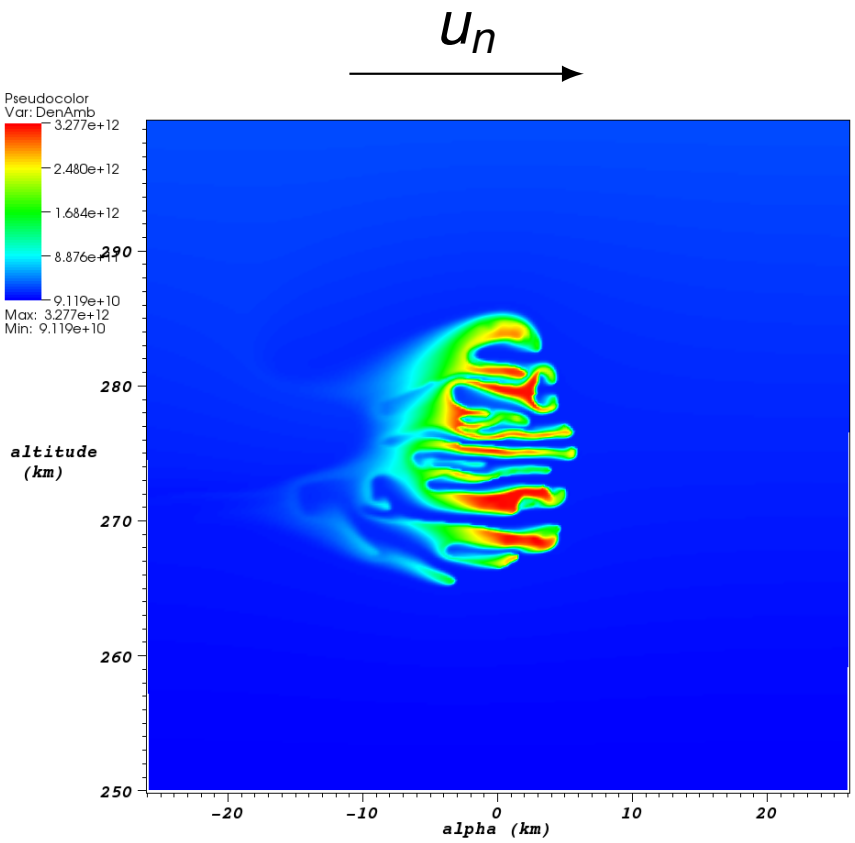


Plasma cloud in magnetic filed

Simulation of the Striation Model

The plasma cloud evolution





We solve the transport equation by method A and method B respectively.

Finite Element Method VS AP Scheme for Elliptic Equation

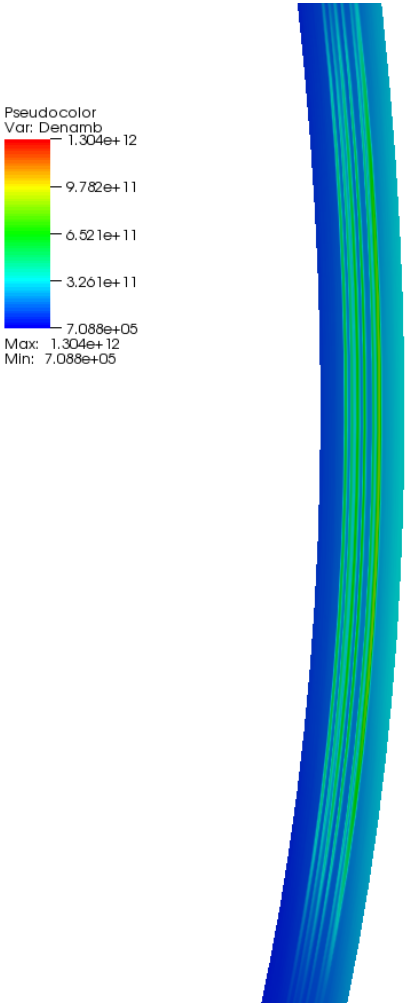
We will compare the numerical results of the 3D anisotropic elliptic equation:

- Test 1
  We take $H_b = 300$km, $H_t = 400$km, the relative difference between the numerical results of the FEM and AP scheme is 10^{-5}.
- Test 2
  We take again $H_b = 300$km, $H_t = 400$km, but we change κ by 10^{-12}, then the AP scheme gives the numerical result comparable to the result of the striation model, whereas the FEM gives a complete wrong result.
- Test 3
  We take $H_b = 1000$km, $H_t = 1200$km,the relative difference between the numerical results of the FEM and AP scheme is 0.13.

Thus, the AP scheme is more robust for anisotropic elliptic equation.

Simulation of the Dynamo Model

Scintillation



Numerical simulation

Conclusions and Perspectives

Conclusions

- The striation model gives the detailed simulation results. But the striation model is valid in high altitude.
- The dynamo model is valid in all layers of ionosphere. However, the dynamo model is costly.

Perspectives

- The coupling model which combines the striation model and the dynamo model
- The adaptive mesh

Thanks for your attention!